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Some Fixed point theorems in Complex Valued Metric Space using Weak Compatible Mappings

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Abstract

The aim of the present paper is to establish fixed point theorems for self mappings under weakly compatibility in Complex valued metric space.

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1 Introduction

An altering distance function is a control function which alters the distance between two points in a metric space. This concept was introduced by Khan, Swaleha and Sessa[7]. Recently altering distance functions have been extended in the context of Menger space by Choudhury and Das[1]. Banach fixed point theorem[2] in a complete metric space has been generalised in many spaces. In 2011,Azam et.al [3] introduced the notion of Complex Valued Metric space and established sufficient conditions for existance of common fixed points of a pair of mappings satisfying a contractive condition.

Recently, Rhoades [10] proved interesting fixed point theorems for $\psi-$ weak contraction in complete metric space. The significance of this kind of contraction can also be derived from the fact that they are strictly relative to famous Banach's fixed point theorems and to some other significant results. Also, motivated by the results of Rhoades and on the lines of Khan et.al. employing the idea of altering distances. The purpose of this paper is to obtain common fixed point of mappings satisfying weakly compatible conditions without exploiting the notion of continuity in setting of complex valued metric space.

The purpose of this paper is to extend the weakly compatibility and occasionally weakly compatibility in Complex Valued Metric space and to obtain fixed point theorems for self mappings satisfying these weakly compatible conditions.

2 Preliminary Notes

Here we recall the definitions, examples and results which will be used in the following section

Definition 2.1.
Let C be the set of complex numbers and let $z, w \in C$.Define a partial order relation $\leq$ on C and z,w $\in$C, $z \leq w$ if and only if Re $z \leq$ Re w and Im $z \leq Im\ w$

Definition 2.2. *Let X be a non-empty set. Suppose that the mapping d: $X \times X \longrightarrow C$ is called a complex valued metric on X if it satisfies the following conditions*

(CM1) $d(x, y) > 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$

(CM2) $d(x,y) = d(y, x)$ *for all* $x,y \in X$

(CM3) $d(x,y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$ *Then(X,d) is called a complex valued metric space*

Example 2.3. *Let X be a Complex valued space.*
Consider $d(z,w) = i|z\text{-}w|$*, Then d is a complex valued metric and (X,d) is a Complex valued metric space.*

Definition 2.4. *Let* (X, d) *be a complex valued metric space*

1. *If* $\{x_n\}$ *be a sequence in X and* $x \in X$*.Then* $\{x_n\}$ *is said to be convergent sequence,if* $\{x_n\}$ *coverges to* $x \in X$ *we denote this by* $\lim_{n\to\infty} x_n = x$*;*

2. *If* $c \in X$ *with* $0 \leq c$ *there exist* $n \in N$ $d(x_n, x_m) \leq c$ *where* $m \in N$*, then* $\{x_n\}$ *is said to be a Cauchy sequence;*

3. *If for every Cauchy sequence in X is convergent then (X,d) is said to be a Complete Complex valued metric space*

Example 2.5. *If* (X, d) *is a metric space then the metric d induces a mapping d:* $X \times X \longrightarrow C$*, defined by* $d(x, y) = |x_1 - x_2| + i|\ y_1 - y_2|$ *for all* $x, y \in X$*. Then* (X, d) *is a Complex Valued Metric space*

Definition 2.6. *Let f and g be self-maps on a set X,if* $w = fx = gx$ *for some* $x \in X$ *, then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g*

Definition 2.7. *Let f and g are self-maps defined on a set X.Then f and g said to be weakly compatible if they commute at their coincidence points.*

Definition 2.8. *Let X be a non-empty set f,g:X $\longrightarrow X$ be mappings.A pair (f,g) is called weakly compatible if $x \in X$ fx=gx implies fgx= gfx*

3 Main Results

Theorem 3.1. *Let (X, d) be a Complex va lued metric space and f,g:X $\longrightarrow X$ be mappings, $\psi : X \longrightarrow X$ is an altering function such that f (X) $\subset g(X)$ for all x,y $\in X$ and saisfy the condition*

$$\psi[d(fx, fy)] \leq \alpha\psi[d(gx, gy)] + \beta\psi[d(fx, gx)] + \gamma\psi[d(fy, gy)] + \delta\psi[d(fx, gy)] \quad (1)$$

and f(X) $\subset g(X)$ is satisfied and f(X) or g(X) is complete,then f and g have a unique point of coincidence. Furthermore if (f , g) is weakly compatible pair and $\rho = \frac{(\alpha+\beta+\gamma+\delta)}{2-(\alpha+\beta+\gamma+\delta)}$ <1 then f,g have unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary we define a sequence y_n such that y_n= fx_{n-1}= gx_n for all $n > 0$ If $y_n = y_{n-1}$ for any n ,then $y_n = y_m$ for all $m > n$ hence $\{y_n\}$ is a cauchy sequence.

If $y_{n-1} \neq y_n$ for all n,then from (1) we have

$$\begin{aligned} \psi[d(fx_n, fx_{n-1})] &\leq \alpha\psi[d(gx_n, gx_{n-1})] + \beta\psi[d(fx_n, gx_n)] + \gamma\psi[d(fx_{n-1}, gx_{n-1})] \\ &\quad +\delta\psi[d(fx_n, gx_{n-1})] \\ \psi[d(y_n, y_{n-1}) &\leq \alpha\psi[d(y_{n-1}, y_{n-2})] + \beta\psi[d(y_n, y_{n-1})] + \gamma\psi[d(y_{n-1}, y_{n-2})] \\ &\quad +\delta\psi[d(y_n, y_{n-2})] \end{aligned}$$

Putting L_n =$d(y_n, y_{n+1})$ We have

$\psi(L_{n-1}) \leq \alpha\ \psi[L_{n-2}]+\beta\ \psi[L_{n-1}]+\gamma\ \psi[L_{n-2}]+\delta\psi[L_{n-1}]$

$$(1-\beta-\delta)\psi[L_{n-1}] \leq (\alpha+\gamma)\psi[L_{n-2}] \tag{2}$$

Using symmetry of (1) in x and y we get

$$(1-\alpha-\gamma)\psi[L_{n-1}] \leq (\beta+\delta)\psi[L_{n-2}] \tag{3}$$

Combining (2) and (3) we get

$\psi[L_{n-1}] \leq \frac{(\alpha+\beta+\gamma+\delta)}{2-(\alpha+\beta+\gamma+\delta)}\ \psi[L_{n-2}] = \rho\ \psi[L_{n-2}]$

and so $\psi[L_{n-1}] \leq (\rho)^{n-1}\ \psi[L_0]$ where $\rho = \frac{(\alpha+\beta+\gamma+\delta)}{2-(\alpha+\beta+\gamma+\delta)} < 1$

If m$>n$ we have

$$\begin{aligned}
\psi[d(y_n,y_m)] &\leq \psi[d(y_n,y_{n+1})]+\psi[d(y_{n+1},y_{n+2})]+ \ldots\ldots\ldots\ldots\ldots + \psi[d(y_{m-1},y_m)] \\
&\leq \psi[L_n]+\psi[L_{n+1}]+ \ldots\ldots\ldots\ldots + \psi[d(L_{m-1})] \\
&\leq \rho^n\psi[L_0]+\rho^{n+1}\psi[L_0]+ \ldots\ldots\ldots\ldots + \rho^{m-1}\psi[L_0] \\
&\leq \frac{\rho^n}{1-\rho}\psi[L_0]
\end{aligned}$$

Since $\rho \leq \|\ \psi[d(y_n,y_m)]\ \| \to 0$ therefore $\psi(d(y_n,y_m) \to 0$ and so $d(y_n,y_m) \to 0$. Hence $\{y_n\}$ is a Cauchy sequence .Thus by the completeness of X $\{y_n\}$ is Convergent and there exist a point z such that $y_n \to z$ as n→∞ Since f(X) is complete and $\{y_n\}$ is a cauchy sequence in f(X) , so $\{y_n\}$ must be convergent in f(X) .

Let $y_n \to z \in \inf(X)$.

Since $z \in f(X) \subset$ g(X), let z= $g(v)$

We show that gv= fv From(1)

$$\begin{aligned}
\psi[d(fv,z)] &\leq \psi[d(fv.fx_n)] + \psi[d(fx_n,z)] \\
&\leq \alpha\psi[d(gv,gx_n)] + \beta\psi[d(fv,gv)] + \gamma\psi[d(fx_n,gx_n)] + \\
&\quad \delta\psi[d(fv,gx_n)] + \psi[d(fx_n,z)] \\
&= \alpha\psi[d(z,y_{n-1})] + \beta\psi[d(fv,z)] + \gamma\psi[d(y_n,y_{n-1})] + \\
&\quad \delta\psi[d(fv,y_{n-1})] + \psi[d(y_n z)] \\
(1-\beta-\delta)\psi[d(fv,z)] &\leq (\alpha+\gamma+\delta)\psi[d(z,y_{n-1})] + (\gamma+1)\psi[d(y_n,z)]
\end{aligned}$$

Since $y_n \to z$. We get $\psi[d(fv,z)] = 0$, So that $d(fv,z) = 0$

That is $fv = z = gv$

Thus z is the coincidence point of f and g. Further f and g is weakly compatible z is the unique fixed point of f and g. □

Example 3.2.

Let (X,d) be a Complex Valued metric space and d(z,w)=i$|z - w|$,where z,w$\in X$. Define $\psi(x)$= x^2 and also define f(x) = $\frac{x}{9}$ and g(x) =$\frac{8x}{9}$.

Also f(0)=0 and g(0)=0 So that fg(0)=gf(0),

$\{f,g\}$ is weakly compatible.So that it satisfies all the conditions of the theorem, so that 0 is the unique common fixed point of f and g.

Theorem 3.3.

Let A,L,M and S be self maps on a Complex valued metric space (X,d) and suppose that the following conditions are satisfied

(1) L(X) $\subset S(X)$, M(X) $\subset A(X)$

(2)One of S(X) or A(X) is a closed subset of X

(3) The pairs (L,A) and (M,S) are weakly compatible

Then for all x,y $\in X$

$$d(Lx, My) \leq min\{d(Sy, My), d(Ax, Lx), d(Ax, My), d(Ax, Sy), d(Sy, Lx)\} \quad (4)$$

.Then A,L,M and S has a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X.
Since L(X)$\subset S(X)$ One can find a point x_1 in X with $LX_0 = Sx_1 = y_0$.
Also M(X) $\subset A(X)$,Choose a point $x_2 \in X$ with $Mx_1 = Ax_2 = y_1$
Inductively we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$Lx_{2n} = Sx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = Ax_{2n+2} = y_{2n+1}$
Put x = x_{2n} and y = x_{2n+1} in (i) We get

$$\begin{aligned}
d(LX_{2n}, My_{2n}) &\leq min\{d(Sx_{2n}, Lx_{2n}), d(Sx_{2n+1}, Lx_{2n}), d(Ax_{2n}, Mx_{2n+1}), d(Ax_{2n}, Sx_{2n+1})\} \\
&\leq min\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1}), d(y_{2n-1}, y_{2n}) \\
&\leq min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\
&\leq min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\
d(y_{2n}, y_{2n+1}) &\leq min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}
\end{aligned}$$

Similarly we have
d(y_{2n},y_{2n+1}) $\leq$ min{d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n})}
Finally d(y_n,y_{n+1}) $\leq$ min{d(y_{n-1},y_n),d(y_n,y_{n+1})}
Now show that $\{y_n\}$ is a Cauchy Sequence.
Consider

$$d(y_n, y_{n+1}) \leq min\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$$

As n$\rightarrow \infty$ Each of these terms d(y_{n-1},y_n) and d(y_n,y_{n+1}) $\rightarrow 0$
That is d(y_n,y_{n+1}) $\rightarrow 0$

therefore $\{y_n\}$ is Cauchy Sequence.

Since X is a Complete metric space,$\{y_n\} \to$z

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Mx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1}$

$= \lim_{n\to\infty} Ax_{2n} =$ z

Suppose that S(X) is closed then for some $\in X$ We have S(v)= z

d(Lx_{2n},Mv)$\leq$ min{d(Ax_{2n},Lx_{2n}),d(Sv,Mv),d(Sv,Lx_{2n}),d(Ax_{2n},Mv),d(Ax_{2n},Sv)}

As n $\to \infty$ We get

d(z,Mv)$\leq$ min{d(z,Mv),d(z,Mv),d(z,z)}

d(z,Mv)$\leq$ min{d(z,Mv),0} d(z,Mv)=0

$Therefore$ Mv = z

M(v)=S(v)=z.

Since (M,S) and is weakly compatible'M(S(v))= S(M(v))

Hence Mz=Sz

Next we have x=x_{2n} and y = z

d(Lx_{2n} ,Mz)$\leq$ min{d(Sz,Mz),d(Ax_{2n},Lx_{2n}),is d(z ,Sz)$\leq$ min{d(z,z),d(Sz,Sz),d(Sz,z),d(z,Sz),d(,Sz)}

d(z,Sz)$\leq$ $min\{d(z, Sz), 0\}$

d(z,Sz)= 0 and Sz=z

Therefore we get Mz=Sz=z

Since A(X)$\subset S(X)$ then there exist w$\in$Xsuch that Aw=Mz=Sz=z

Put x=w and y=z in (i) we get

d(Lw,Mz)$\leq$ $min\{d(Sz, Mz), d(Az, Lz), d(Az, Mw), d(Az, Sw), d(Sw, Lz)\}$

and so we have d(Lw,z))$\leq$ $min\{d(z, z), d(z, Lw), d(z, z), d(Az, Lw), d(z, z)\}$

d(Lw,z)$\leq$min{d(z,Lw),0}

ie d(Lw,z)= 0 and so Lw=z

Therefore Lw=Aw=z

Also it is given that (L,A) is weakly compatible L(A(w))= A(L(w))

That is We get Lz=Az

Now put x=z and y=x_{2n+1} in (i) we get

d(Lz,Mx_{2n+1})$\leq$ min{d(Sx_{2n+1},Mx_{2n+1}),d(Az,Lz),d(Az,Mx_{2n+1}),d(Az,Sx_{2n+1}),d(Sx_{2n+1},Lz)}

As n$\to\infty$ d(Lz,z)$\leq$ min{d(z,Lz),d(z,z),d(z,Lz),d(z,z),d(z,Lz)}

ie we get d(z,Lz)$\leq$ 0 and so d(z,Lz) =0

Hence Lz=z and combine all these results we get

z=Az=Mz=Sz,Thus z is the common fixed point.

To prove uniqueness

Let u be another fixed point of A,L,M and S.Taking x=z and y=u we get

$$\begin{aligned} d(Lz,Mu) &\leq min\{d(Su,Mu),d(Az,Lz),d(Az,Mu),d(Az,Su),d(Su,Lz) \\ &\leq min\{d(z,z),d(u,z),d(u,u),d(z,u),d(z,u)\} \end{aligned}$$

That is d(z,u)=0 and hence d(Lz,Mu)=0

Therefore z=u ,the fixed point is unique.

□

Example 3.4.

Let (X,d) be a Complex valued metric space and d(z,w)= i$|z-w|$,where z,w$\in X$.Define

$$A(x) = \begin{cases} 0,\ x=0 \\ 1/2-x,\ 0<x\leq 1/2 \\ 1+x,\ 1/2<x\leq 1 \end{cases} \quad S(X) = \begin{cases} 0,\ x=0 \\ 1/2-x,\ 0<x\leq 1/2 \\ \frac{2-x}{2},\ 1/2<x\leq 1 \end{cases}$$

$$L(X) = \begin{cases} 0,\ x=0 \\ 1/2,\ 0<x\leq 1 \end{cases} \quad M(x) = \begin{cases} 0,\ x=0 \\ 1,\ 0<x\leq 1 \end{cases}$$

A,L,M and S satisfies all the conditions of the theorem and have a unique common *fixed point 0 ∈X .In this example L and A commute at their coincidence point 0 ∈ X .So L and A are weakly compatible. Similarly M and S are weakly compatible.*

Definition 3.5. *Two selfmappings f and g of a Complex Valued Metric space are occasionally weakly compatible if and only if there is a point $x \in X$ which is a coinidence point of f and g at which f and g commute.*

Theorem 3.6. *Let (X,d) be a Complex Valued metric space. Also (L,A) and (M,S) are occasionally weakly compatible maps in X satisfying*
$Min\{d(Lx, My), d(Sy, Lx)\} \leq \alpha\ d(Ax,Lx) + \beta\ d(Ax,Sy)$ *..........(1) for all* $x, y \in X$
0< $\alpha,\beta < 1$ *such that* $\alpha+\beta > 1$
Then L,A,M and S have a unique common fixed point.

Proof: Since the pairs (L,A) and weakly (M,S) are occasionally weakly compatible,there exist points $u, v \in X$ such that Lu=Au, LAu=ALu and Mv=Sv,MSv=SMv

Now we show that Lu=Mv
By putting x=u and y=v in (1), then we get

$$\begin{aligned} min\{d(Lu, Mv), d(Sv, Lu)\} &\leq \alpha d(Au, Lu) + \beta d(Au, Sv)\} \\ d(Lu, Mv) &\leq \alpha d(Au, Lu) + \beta d(Au, Sv)\} \\ d(Lu, Mv) &\leq \alpha d(Au, Lu) + \beta d(Au, Sv) \\ d(Lu, Mv) &\leq \beta d(Lu, Mv)\} \\ (1-\beta)d(Lu, Mv) &\leq 0 \\ Since 0 < \beta < 1, We\ have\ d(Lu, Mv) = 0 \end{aligned}$$

Thus
we have Lu=Mv.Threrefore Lu=Mv=Au=Sv
Moreover if there is another point z such that Lz=Az . By using (1) we have

Lz=Az=Mv=Sv or Lu=Lz
Hence w=Lu=Au is the fixed point of coincidence of L and A and w is the unique common fixed point of L and A.
Similarly there is a unique point z $\in X$ such that z = Mz = Sz.Suppose that w$\neq z$ by taking x=w and y=z in (1) we get

$$\begin{aligned} Min\{d(Lw, Mz), d(Sw, Lz)\} &\leq \alpha d(Aw, Lw) + \beta d(Aw, Sz) \\ min\{d(w, z), d(z, w)\} &\leq \alpha d(w, w) + \beta d(w, z) \\ d(w, z)\} &\leq \beta d(w, z) \\ (1 - \beta)d(w, z) &\leq 0 \; and \; therefore \; d(w, z) = 0. \end{aligned}$$

Thus we have w=z, w is the unique common fixed point of L,A,M and S in X. □

Example 3.7.
Let (X,d) be a Complex valued metric space and d(z,w)= $i|z - w|$*,where z,w*$\in X$*. Define*

$$A(x) = \begin{cases} 0, \; x = 0 \\ 1/2 - x, \; 0 < x \leq 1/2 \\ 1 + x, \; 1/2 < x \leq 1 \end{cases} \quad S(x) = \begin{cases} 0, \; x = 0 \\ 3/2 - x, \; 0 < x \leq 1/2 \\ x, \; 1/2 < x \leq 1 \end{cases}$$

Define

$$L(x) = \begin{cases} 0, \; x = 0 \\ 2 \; 0 < x \leq 2 \end{cases} \quad M(x) = \begin{cases} 0, \; x = 0 \\ 1 \; 0 < x \leq 2 \end{cases}$$

A,L,M ,S satisfies all the conditions of the theorem and have a common fikxed point 0 $\in X$

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