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The Use of Determinants for Constructing Curves and Surfaces through Specified Points

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Introduction

We describe a technique that uses determinants to construct lines, circles, and general conic sections through specified points in the plane. The procedure is also used to pass planes and spheres in 3-space through fixed points.

THEOREM: A homogeneous linear system with as many equations as unknowns has a nontrivial solution if and only if the determinant of the coefficient matrix is zero.

We will now show how this result can be used to determine equations of various curves and surfaces through specified points.

A Line Through Two Points: Suppose that (x_1, y_1) and (x_2, y_2) are two distinct points in the plane. There exists a unique line

$$c_1x + c_2y + c_3 = 0 \quad (1)$$

that passes through these two y points (Figure 1).

Note that $c_1, c_2,$ and c_3 are not all zero and that these coefficients are unique only up to a multiplicative constant. Because (x_1, y_1) and (x_2, y_2) lie on the line, substituting them in (1) gives the two equations

$$c_1x_1 + c_2y_1 + c_3 = 0 \quad (2)$$

$$c_1x_2 + c_2y_2 + c_3 = 0 \quad (3)$$

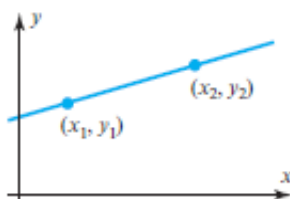


Figure 1

The three equations, (1), (2), and (3), can be grouped together and rewritten as

$$xc_1 + yc_2 + c_3 = 0$$

$$x_1c_1 + y_1c_2 + c_3 = 0$$

$$x_2c_1 + y_2c_2 + c_3 = 0$$

which is a homogeneous linear system of three equations for $c_1, c_2,$ and c_3 . Because $c_1, c_2,$ and c_3 are not all zero, this system has a nontrivial solution, so the determinant of the coefficient matrix of the system must be zero.

That is,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad (4)$$

Consequently, every point (x, y) on the line satisfies (4); conversely, it can be shown that every point (x, y) that satisfies (4) lies on the line.

Equation of a Line

Find the equation of the line that passes through the two points $(2, 1)$ and $(3, 7)$.

Solution: Substituting the coordinates of the two points into Equation (4) gives

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{vmatrix} = 0$$

The cofactor expansion of this determinant along the first row then gives

$$-6x + y + 11 = 0$$

A Circle through Three Points: Suppose that there are three distinct points in the plane, $(x_1, y_1), (x_2, y_2),$ and $(x_3, y_3),$ not all lying on a straight line.

From analytic geometry we know that there is a unique circle, say,

$$c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0 \tag{5}$$

that passes through them (Figure 2).

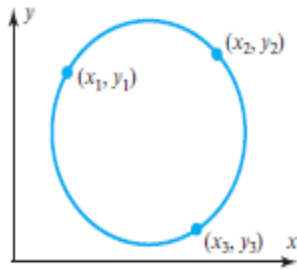


Figure 2

Substituting the coordinates of the three points into this equation gives

$$c_1(x_1^2 + y_1^2) + c_2x_1 + c_3y_1 + c_4 = 0 \tag{6}$$

$$c_1(x_2^2 + y_2^2) + c_2x_2 + c_3y_2 + c_4 = 0 \tag{7}$$

$$c_1(x_3^2 + y_3^2) + c_2x_3 + c_3y_3 + c_4 = 0 \tag{8}$$

As before, Equations (5) through (8) form a homogeneous linear system with a nontrivial solution for $c_1, c_2, c_3,$ and c_4 . Thus the determinant of the coefficient matrix is zero:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{9}$$

This is a determinant form for the equation of the circle.

Equation of a Circle

Find the equation of the circle that passes through the three points (1, 7), (6, 2), and (4, 6).

Solution: Substituting the coordinates of the three points into Equation (9) gives

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 50 & 1 & 7 & 1 \\ 40 & 6 & 2 & 1 \\ 52 & 4 & 6 & 1 \end{vmatrix} = 0$$

which reduces to

$$10(x^2 + y^2) - 20x - 40y - 200 = 0$$

In standard form this is

$$(x - 1)^2 + (y - 2)^2 = 5^2$$

Thus the circle has center (1, 2) and radius 5.

A General Conic Section Through Five Points:

Suppose that there are five distinct points in the plane, $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and (x_5, y_5) . The general equation of a conic section in the plane (a parabola, hyperbola, or ellipse, or degenerate forms of these curves) is given by

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0$$

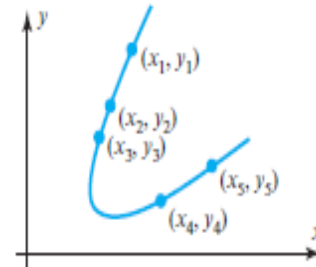


Figure 3

This equation contains six coefficients, but we can reduce the number to five if we divide through by any one of them that is not zero. Thus only five coefficients must be determined, so five distinct points in the plane are sufficient to determine the equation of the conic section (Figure 3). As before, the equation can be put in determinant form:

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0 \tag{10}$$

Equation of an Orbit

An astronomer who wants to determine the orbit of an asteroid about the Sun sets up a Cartesian coordinate system in the plane of the orbit with the Sun at the origin. Astronomical units of measurement are used along the axes (1 astronomical unit = mean distance of Earth to Sun = 93 million miles). By Kepler’s first law, the orbit must be an ellipse, so the astronomer makes five observations of the asteroid at five different times and finds five points along the orbit to be (8.025, 8.310), (10.170, 6.355), (11.202, 3.212), (10.736, 0.375), (9.092, -2.267)

Find the equation of the orbit.

Solution: Substituting the coordinates of the five given points into (10) and rounding to three decimal places give

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 64.401 & 66.688 & 69.056 & 8.025 & 8.310 & 1 \\ 103.429 & 64.630 & 40.386 & 10.170 & 6.355 & 1 \\ 125.485 & 35.981 & 10.317 & 11.202 & 3.212 & 1 \\ 115.262 & 4.026 & 0.141 & 10.736 & 0.375 & 1 \\ 82.664 & -20.612 & 5.139 & 9.092 & -2.267 & 1 \end{vmatrix} = 0$$

The cofactor expansion of this determinant along the first row yields

$$386.802x^2 - 102.895xy + 446.029y^2 - 2476.443x - 1427.998y - 17109.375 = 0$$

Figure 4 is an accurate diagram of the orbit, together with the five given points.

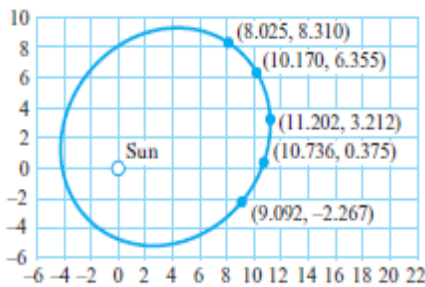


Figure 4

A Plane Through Three Points: The plane in 3-space with equation

$$c_1x + c_2y + c_3z + c_4 = 0$$

that passes through three noncollinear points (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) is given by the determinant equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \tag{11}$$

Equation of a Plane

The equation of the plane that passes through the three noncollinear points $(1, 1, 0)$, $(2, 0, -1)$, and $(2, 9, 2)$ is

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0$$

which reduces to

$$2x - y + 3z - 1 = 0$$

A Sphere Through Four Points: The sphere in 3-space with equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0$$

that passes through four noncoplanar points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and

(x_4, y_4, z_4) is given by the following determinant equation:

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \tag{12}$$

Equation of a Sphere

The equation of the sphere that passes through the four points $(0, 3, 2)$, $(1, -1, 1)$, $(2, 1, 0)$, and $(5, 1, 3)$ is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 0 & 3 & 2 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 1 \\ 35 & 5 & 1 & 3 & 1 \end{vmatrix} = 0$$

This reduces to:

$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0$$

Which in standard form is

$$(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 9$$

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