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Nash - Equilibrium solutions for Fuzzy Rough Continuous Static Games

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Abstract
Fuzzy rough is used to measure the various uncertainties with high vagueness and imprecision in continuous static games. In this article we develop a new class defined as fuzzy rough continuous static games (FRGSG) in which the objective functions and the constraints have fuzzy rough nature. The fuzzy part is converted into deterministic ones by utilizing the α-cut methodology and the roughness part is transformed into two problems corresponding to the upper and lower approximation models thus, the result is four cases. The necessary conditions for determining Nash- Equilibrium points of fuzzy rough continuous static games are achieved and a numerical example is performed to support the theoretical claims.
Keywords: *Continuous static game, Game theory, Fuzzy sets, Rough set theory, linear Programming and Non-linear Programming*

1 Introduction

Game theory is a valuable tool for decision-making in the conflict of interests between decision-makers to choose the best joint approach by choosing the best desirability in general. Game theory has broad applications in the fields of social life, economics, policy, engineering, sciences, biology, etc.[6],[14] Most real world decision-making problems need to be modeled as problems of vector optimization (VOPs, Continuous static games are another forms of VOPs by considering the more general state of different decision-makers, each with its cost criterion[[5],[4]].This generalization proposes the competition possibility between the system controls, called "players," and therefore the optimization problem under consideration is called a "game".Every player in the game controls a given system parameter sub-set (called its control vector) and tries to reduce its own scalar cost criterion, subject to specified constraints [3],[17]. In real-life problems including rough variables, logical variables, fuzzy numbers and random variables, etc., uncertainty is common.Zadeh investigated the Fuzzy Sets concept[8]. Different researchers have recently done a fuzzy and rough set theory combination. Rough set theory (RST) can be considered as a new mathematical tool for incomplete data analysis introduced by Pawlak [[11],[16]]to deal with uncertainty. It has several applications found in various fields such as decision support, engineering, environment, medicine [[18],[1],[2]]. It communicates vagueness by utilizing a limit district of a set not through membership function like a fuzzy set[[15],[7]].See [[8].[10]] for details. The paper is arranged as follows: Following introduction, Section 2 introduces some preliminaries and notions . In the next section mathematical formulation of FRCSG was presented with the definitions for different types of the solution sets . Section 4 incorporates the Nash - Equilibrium solutions for the Upper - Upper Game (UUG) and step algorithm is explained . A numerical example is worked out in Section 5. Finally, some conclusions are presented in Section 6.

2 Notations

In this section, we provide some basic definitions required through our article

Definition 2.1 Let R be the set of real numbers, the fuzzy number $\tilde{a}$ is a mapping $\mu_{\tilde{a}}: R \longrightarrow [0,1]$ with the following properties[9]:

1. $\mu_{\tilde{a}}$ is upper semi continuous membership function.
2. $\tilde{a}$ is convex fuzzy set, i.e.

$$\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min[\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)] \forall x, y \in R, \lambda \in [0,1]$$

3. $\tilde{a}$ is normal i.e.,$\exists x_0 \in R$ for which $\mu_{\tilde{a}}(x_0) = 1$
4. $supp\tilde{a} = x \in R: \mu_{\tilde{a}}(x) > 0$is a support of the $\tilde{a}$.

Definition 2.2 The $\alpha -$ level set of the fuzzy parameters $\tilde{a}$, is an ordinary set $L_\alpha(\tilde{a})$ for which the degree of its membership function exceeds the level set $\alpha \in [0,1]$ where [[12],[13]]:

$$L_\alpha(\tilde{a}) = \{a \in R^m | \mu_{\tilde{a}} \geq \alpha\} = \{a \in [\tilde{a}_\alpha{}^L, \tilde{a}_\alpha{}^U] | \mu_{\tilde{a}} \geq \alpha\} \quad (1)$$

Definition 2.3 Let W be the universal set, R be the equivalence relation on W, $[w]_R$ be the set of equivalence class of R, and V be a non-empty subset of W. The upper and lower approximations of the set Vare defined as[[20],[21]].:

$$\overline{RV} = \{w \in W | [w]_R \cap V \neq \phi\}$$

$$\underline{RV} = \{w \in W | [w]_R \subseteq V\}$$

$V = \overline{RV} - \underline{RV}$ If $V \neq \overline{RV} - \underline{RV}$ then set Vis called a rough set.

Definition 2.4 [19] Let S denote a compact set of real numbers. A **fuzzy rough interval** $\tilde{S}^R$ is defined as $\tilde{S}^R = [\tilde{S}^L, \tilde{S}^U]$ where $\tilde{S}^L$ and $\tilde{S}^U$ are fuzzy sets called lower and upper approximation fuzzy of $\tilde{S}^R$ with $\tilde{S}^L \subseteq \tilde{S}^U$

3 Game Formulation

The continuous static games with fuzzy rough cost functions and fuzzy rough conditions is given by:

$$min\tilde{G_i}^R(\tilde{x}^R, \tilde{u_i}^R) \tag{2}$$

S.T.

$$\tilde{g}^R(\tilde{x}^R, \tilde{u_i}^R) = 0 \tag{3}$$

$$\tilde{h}^R(\tilde{x}^R, \tilde{u_i}^R) \geq 0 \tag{4}$$

Where, $\tilde{u_i}^R$,$\tilde{G_i}^R$ are fuzzy rough controls ,cost functions for players respectively, and $\tilde{x}^R$ is a fuzzy rough state.

This game 2 - 4 can be written as:

$$min[\tilde{G_i}^L(\tilde{x}^L, \tilde{u_i}^L), \tilde{G_i}^U(\tilde{x}^U, \tilde{u_i}^U)] \tag{5}$$

S.T.

$$[\tilde{g}^L(\tilde{x}^L, \tilde{u_i}^L), \tilde{g}^U(\tilde{x}^U, \tilde{u_i}^U)] = 0 \tag{6}$$

$$[\tilde{h}^L(\tilde{x}^L, \tilde{u_i}^L), \tilde{h}^U(\tilde{x}^U, \tilde{u_i}^U)] \geq 0 \tag{7}$$

Now, the game 5 - 7 can be divided into two games

1. The lower game

$$min\tilde{G_i}^L(\tilde{x}^L, \tilde{u_i}^L) \tag{8}$$

S.T.

$$\tilde{g}^L(\tilde{x}^L, \tilde{u_i}^L) = 0 \tag{9}$$

$$\tilde{h}^L(\tilde{x}^L, \tilde{u_i}^L) \geq 0 \tag{10}$$

2. The upper game

$$min\tilde{G_i}^U(\tilde{x}^U, \tilde{u_i}^U) \tag{11}$$

S.T.

$$\tilde{g}^U(\tilde{x}^U, \tilde{u_i}^U) = 0 \tag{12}$$

$$\tilde{h}^U(\tilde{x}^U, \tilde{u_i}^U) \geq 0 \tag{13}$$

By using α - level, $\alpha \in [0,1]$, We should restructure the game 8 - 13 to the following:

1. The lower game

$$min[G_{i_\alpha}{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}), G_{i_\alpha}{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL})] \quad (14)$$

S.T.

$$[g_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}), g_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) = 0 \quad (15)$$

$$[h_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}), h_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL})] \geq 0 \quad (16)$$

2. The upper game

$$min[G_{i_\alpha}{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}), G_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU})] \quad (17)$$

S.T.

$$[g_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}), g_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) = 0 \quad (18)$$

$$[h_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}), h_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU})] \geq 0 \quad (19)$$

The following four cases of FRCSG are formed from the above decomposition game as follows:

1. **Case 1:** Upper - Upper Game (UUG)

$$minG_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) \quad (20)$$

S.T.

$$g_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) = 0 \quad (21)$$

$$h_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) \geq 0 \quad (22)$$

2. **Case 2 :** Lower - Upper Game (LUG)

$$minG_{i_\alpha}{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}) \quad (23)$$

S.T.

$$g_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}) = 0 \quad (24)$$

$$h_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}) \geq 0 \quad (25)$$

3. **Case 3 :** Upper - Lower Game (ULG)

$$minG_{i_\alpha}{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) \quad (26)$$

S.T.

$$g_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) = 0 \quad (27)$$

$$h_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) \geq 0 \quad (28)$$

4. **Case 4 :** Lower - Lower Game (LLG)

$$minG_{i_\alpha}{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}) \quad (29)$$

S.T.

$$g_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}) = 0 \quad (30)$$

$$h_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}) \geq 0 \quad (31)$$

3.1 The types of solution sets for the game

• The optimal solution set O for (UUG) is

$$O = \{U_{i_\alpha}{}^{UU} \in E^r | g_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) = 0, h_\alpha{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) \geq 0,$$
$$G_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, U_{i_\alpha}{}^{UU}) = minG_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU})\} \quad (32)$$

(a) If $O_1 = \{O \cap g_\alpha{}^{LU} \cap h_\alpha{}^{LU} \cap g_\alpha{}^{UL} \cap h_\alpha{}^{UL} \cap g_\alpha{}^{LL} \cap h_\alpha{}^{LL}\} \neq \phi$ then, O_1 called the **α -surely optimal solution set** and $O \sim O_1$ called the **α -possibly optimal solution set**.

(b) If $O_1 = \phi$ then, the game doesn't have a α -surely optimal solution set.

(c) If $O_2 = \{O \cap g_\alpha{}^{LU} \cap h_\alpha{}^{LU}\} \neq \phi$ then, O_2 called the **α -surely upper optimal solution set** and $O_1 \sim O_2$ called the **α -possibly upper optimal solution set**.

(d) If $O_2 = \phi$ then, O_1 called the **α - Upper Upper optimal solution set** and

$$O_3 = \{U_{i_\alpha}{}^{LU} \in E^r | g_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}) = 0, h_\alpha{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU}) \geq 0,$$
$$G_{i_\alpha}{}^{LU}(x_\alpha{}^{LU}, U_{i_\alpha}{}^{LU}) = minG_{i_\alpha}{}^{LU}(x_\alpha{}^{LU}, u_{i_\alpha}{}^{LU})\} \quad (33)$$

called the **α - Lower Upper optimal solution set.**

• The optimal solution set O_4 for (ULG) is

$$O_4 = \{U_{i_\alpha}{}^{UL} \in E^r | g_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) = 0, h_\alpha{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL}) \geq 0,$$

$$G_{i_\alpha}{}^{UL}(x_\alpha{}^{UL}, U_{i_\alpha}{}^{UL}) = minG_{i_\alpha}{}^{UL}(x_\alpha{}^{UL}, u_{i_\alpha}{}^{UL})\} \tag{34}$$

(a) If $O_5 = \{O_4 \cap g_\alpha{}^{LL} \cap h_\alpha{}^{LL}\} \neq \phi$ then, O_5 called the $\boldsymbol{\alpha}$ **-surely Lower optimal solution set** and $O_4 \sim O_5$ called the $\boldsymbol{\alpha}$ **-possibly Lower optimal solution set**.

(b) If $O_5 = \phi$ then, O_4 called the $\boldsymbol{\alpha}$ **- Upper Lower optimal solution set** and

$$O_6 = \{U_{i_\alpha}{}^{LL} \in E^r | g_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}) = 0, h_\alpha{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL}) \geq 0,$$

$$G_{i_\alpha}{}^{LL}(x_\alpha{}^{LL}, U_{i_\alpha}{}^{LL}) = minG_{i_\alpha}{}^{LL}(x_\alpha{}^{LL}, u_{i_\alpha}{}^{LL})\} \tag{35}$$

called the $\boldsymbol{\alpha}$ **- Lower Lower optimal solution set.**

Definition 3.1 If $O_1 \neq \phi$ then O_1 contains all α - surely optimal solutions, hence it is called the α- surely optimal set.

Definition 3.2 $O \sim O_1$ contains α -possibly optimal solutions,hence it is called the α -possibly optimal set.

Definition 3.3 If $O_2 \neq \phi$ then O_2 contains all α - surely upper optimal solutions, hence it is called the α-surely upper optimal set.

Definition 3.4 $O_1 \sim O_2$ contains α -possibly upper optimal solutions,hence it is called the α -possibly upper optimal set.

Definition 3.5 If $O_2 = \phi$ then O_1 contains α - upper upper optimal solutions, hence it is called the α- upper upper optimal set and O_3 is called the α - lower upper optimal set.

Definition 3.6 If $O_5 \neq \phi$ then O_5 contains all α - surely lower optimal solutions, hence it is called the α-surely lower optimal set.

Definition 3.7 $O_4 \sim O_5$ contains α -possibly lower optimal solutions,hence it is called the α -lower upper optimal set.

Definition 3.8 If $O_5 = \phi$ then O_4 contains α - upper lower optimal solutions, hence it is called the α- upper lower optimal set and O_6 is called the α - lower lower optimal set.

4 Nash - Equilibrium solutions for (UUG)

If the coalitions between players are not possible, it is considered the best solution. It is assumed that the Players behave independently, without any of the other players participating, and every player seeks to minimize his cost function.

Definition 4.1 A point $\hat{u}_\alpha{}^{UU} \in \Omega$ is a α - upper upper Nash equilibrium point if and only if for each $i = 1, \dots, r$

$$G_{i_\alpha}{}^{UU}[\zeta(\hat{u}_\alpha{}^{UU}), \hat{u}_\alpha{}^{UU}] \le G_{i_\alpha}{}^{UU}[\zeta(u_{i_\alpha}{}^{UU}, \hat{v}_\alpha{}^{UU}), u_{i_\alpha}{}^{UU}, \hat{v}_\alpha{}^{UU}] \quad (36)$$

For all $u_{i_\alpha}{}^{UU} \in U_{i_\alpha}{}^{UU}$ where, $\hat{u}_\alpha{}^{UU} = (\hat{u}_{i_\alpha}{}^{UU}, \hat{v}_\alpha{}^{UU})$,

$$U_{i_\alpha}{}^{UU} = \{u_{i_\alpha}{}^{UU} \in E^s | h_\alpha{}^{UU}[\zeta(u_{i_\alpha}{}^{UU}, v_\alpha{}^{UU}), u_{i_\alpha}{}^{UU}, v_\alpha{}^{UU}] \ge 0\} \quad (37)$$

and $x_\alpha{}^{UU} = \zeta(u_\alpha{}^{UU})$ is the solution to 21. For a local α - upper upper Nash equilibrium point replace $U_{i_\alpha}{}^{UU}$ by $B_{i_\alpha}{}^{UU} \cup U_{i_\alpha}{}^{UU}$ for some ball $B_{i_\alpha}{}^{UU} \subset E^s$ centered at $\hat{u}_{i_\alpha}^{UU}$.

Lemma 4.1 If $\hat{u}_\alpha{}^{UU} \in \Omega$ is a local α - upper upper Nash equilibrium point for the game 20 - 22, and if $\hat{x}_\alpha{}^{UU} = \zeta(\hat{u}_\alpha{}^{UU})$ is the solution to 21, then for each $i = 1, \dots, r$ there exists a vector $\gamma_{i_\alpha}{}^{UU} \in E^n$ defined by

$$\frac{\partial J_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \gamma_{i_\alpha}{}^{UU}]}{\partial x_\alpha{}^{UU}} = 0 \quad (38)$$

$$\frac{\partial J_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \gamma_{i_\alpha}{}^{UU}]}{\partial u_{i_\alpha}{}^{UU}} e^i = 0 \quad (39)$$

For all $e^i \in T_{i_\alpha}{}^{UU}$ where,

$$J_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \gamma_{i_\alpha}{}^{UU}] = G_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, u_\alpha{}^{UU}) - \gamma_{i_\alpha}{}^{UU^T} g_\alpha{}^{UU}(x_\alpha{}^{UU}, u_\alpha{}^{UU}) \quad (40)$$

and the tangent cone to $U_{i_\alpha}{}^{UU}$ at $u_{i_\alpha}{}^{UU}$ ($T_{i_\alpha}{}^{UU}$) is given by

$$T_{i_\alpha}{}^{UU} = \{e^i \in E^s | [\frac{\partial \hat{h}_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}} - \frac{\partial \hat{h}_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}} [\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}]^{-1} \frac{\partial g_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}}] e^i\} \ge 0 \quad (41)$$

where, $\hat{h}_\alpha{}^{UU}$ denotes the active inequality constraints at $u_\alpha{}^{UU} = (u_{i_\alpha}{}^{UU}, v_\alpha{}^{UU})$.

Necessary conditions usable for determining Nash- Equilibrium points may now be stated in the following theorem.

Theorem 4.1 If $\hat{u}_\alpha{}^{UU} \in \Omega$ is a completely regular local α - upper upper Nash equilibrium point for the game 20-22 and $\hat{x}_\alpha{}^{UU} = \zeta(\hat{u}_\alpha{}^{UU})$ is the solution to $g_\alpha{}^{UU}(x_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}) = 0$, then for each $i = 1, \dots, r$ there exist a vector $\lambda_{i_\alpha}{}^{UU} \in E^n$ and a vector $\mu_{i_\alpha}{}^{UU} \in E^q$ such that

$$\frac{\partial L_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \lambda_{i_\alpha}{}^{UU}, \mu_{i_\alpha}{}^{UU}]}{\partial x_\alpha{}^{UU}} = 0 \qquad (42)$$

$$\frac{\partial L_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \lambda_{i_\alpha}{}^{UU}, \mu_{i_\alpha}{}^{UU}]}{\partial u_{i_\alpha}{}^{UU}} = 0 \qquad (43)$$

$$g_\alpha{}^{UU}(\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}) = 0 \qquad (44)$$

$$\mu_{i_\alpha}{}^{UU^T} h_\alpha{}^{UU}(\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}) = 0 \qquad (45)$$

$$h_\alpha{}^{UU}(\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}) \geq 0 \qquad (46)$$

$$\mu_{i_\alpha}{}^{UU} \geq 0 \qquad (47)$$

Where,

$$L_{i_\alpha}{}^{UU}[x_\alpha{}^{UU}, u_\alpha{}^{UU}, \lambda_{i_\alpha}{}^{UU}, \mu_{i_\alpha}{}^{UU}] = G_{i_\alpha}{}^{UU}(x_\alpha{}^{UU}, u_{i_\alpha}{}^{UU}) - \lambda_{i_\alpha}{}^{UU^T} g_\alpha{}^{UU}(x_\alpha{}^{UU}, u_\alpha{}^{UU})$$
$$-\mu_{i_\alpha}{}^{UU^T} h_\alpha{}^{UU}(x_\alpha{}^{UU}, u_\alpha{}^{UU}) \qquad (48)$$

Proof. For each $i = 1, \dots, r$ consider the cone

$$K_{i_\alpha}{}^{UU} = \{y_\alpha{}^{UU} \in E^s | y_\alpha{}^{UU^T} = \{\mu_{i_\alpha}{}^{UU^T} [\frac{\partial h_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}} - \frac{\partial h_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}} [\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}]^{-1} \frac{\partial g_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}}]$$
$$, \mu_{i_\alpha}{}^{UU^T} h_\alpha{}^{UU} = 0, \mu_{i_\alpha}{}^{UU} \geq 0\} \qquad (49)$$

and its polar

$$K^*{}_{i_\alpha}{}^{UU} = \{Z_\alpha{}^{UU} \in E^s | y_\alpha{}^{UU^T} Z_\alpha{}^{UU} \geq 0 \forall y_\alpha{}^{UU} \in K_{i_\alpha}{}^{UU}\} \qquad (50)$$

where all quantities are evaluated at $(\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU})$.Since $\hat{u}_{i_\alpha}{}^{UU}$ is a regular point of $u_{i_\alpha}{}^{UU}$ the tangent cone $T_{i_\alpha}{}^{UU}$ to $U_{i_\alpha}{}^{UU}$ is given by

$$T_{i_\alpha}{}^{UU} = K^*{}_{i_\alpha}{}^{UU} \qquad (51)$$

From this result and lemma 4.1 we have

$$\frac{\partial J_{i_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \gamma_{i_\alpha}{}^{UU}]}{\partial u_{i_\alpha}{}^{UU}} \in K_{i_\alpha}{}^{UU} \tag{52}$$

Thus from 49 we have

$$\frac{\partial J_{i_\alpha}{}^{UU}}{\partial u_{i_\alpha}{}^{UU}} = \{\mu_{i_\alpha}{}^{UU^T}[\frac{\partial h_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}} - \frac{\partial h_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}[\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}]^{-1}\frac{\partial g_\alpha{}^{UU}}{\partial u_{i_\alpha}{}^{UU}}] \tag{53}$$

where $\mu_{i_\alpha}{}^{UU}$ satisfies 45 - 47 .Define

$$\lambda_{i_\alpha}{}^{UU^T} = \gamma_{i_\alpha}{}^{UU^T} - \mu_{i_\alpha}{}^{UU^T}\frac{\partial h_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}[\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}]^{-1} \tag{54}$$

where $\gamma_{i_\alpha}{}^{UU}$ is defined by 40 and

$$\frac{\partial J_{i_\alpha}{}^{UU}}{\partial x_\alpha{}^{UU}} = 0 \tag{55}$$

combining 55 and 54 yields

$$0_\alpha{}^{UU} = \frac{\partial J_{i_\alpha}{}^{UU}}{\partial x_\alpha{}^{UU}} = \frac{\partial G_{i_\alpha}{}^{UU}}{\partial x_\alpha{}^{UU}} - [\lambda_{i_\alpha}{}^{UU^T} + \mu_{i_\alpha}{}^{UU^T}\frac{\partial h_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}[\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}}]^{-1}]\frac{\partial g_\alpha{}^{UU}}{\partial x_\alpha{}^{UU}} \tag{56}$$

which is equivalent to 42. Combining 40, 53 and 54 yields 43, which establishes the theorem.

Similarly, we can prove the necessary condition of the Lower Upper Game (LUG) 23-25,Upper Lower Game (ULG) 26 -3 and Lower Lower Game (LLG) 29 - 31.

The algorithm for the Nash-Equilibrium solutions of fuzzy rough continuous static games follows as:

1. For player i, formulate the upper-upper game (UUG) and its lagrange function.
2. Examine the necessary conditions to obtain α -upper upper nash equilibrium point.
3. If α -upper upper nash equilibrium point is the α surely optimal point , stop otherwise go to step 4.
4. If it is the α - surely upper optimal point go to step 6 otherwise go to step 5
5. Formulate the lower-upper game (LUG) ,its lagrange function and examine the necessary conditions to obtain α - Lower upper optimal point.
6. Formulate the upper - lower game (ULG) ,its lagrange function and examine the necessary conditions to obtain α - upper lower nash equilibrium point.
7. If the α - upper lower nash equilibrium point is the α - surely lower optimal point, stop otherwise go to step 8.
8. Formulate the lower - lower game (LLG) ,its lagrange function and examine the necessary conditions to obtain α -lower lower optimal point.

5 Numerical example

Consider the two player fuzzy rough game with

$$\tilde{G}_1{}^R = \tilde{3}^R \tilde{u}^R \tilde{v}^R - \tilde{1}^R$$

$$\tilde{G}_2{}^R = \tilde{4}^R \tilde{u}^R - \tilde{1}^R (\tilde{v}^R)^2$$

S.T.

$$\tilde{1}^R \tilde{u}^R + \tilde{1}^R \tilde{v}^R - \tilde{2}^R \geq 0$$

In this example we assume

$$\tilde{1}^R = [(\alpha+1, -\alpha+2); (2\alpha+1, -2\alpha+3)]$$

$$\tilde{2}^R = [(\alpha+2, -\alpha+3); (2\alpha+2, -2\alpha+4)]$$

$$\tilde{3}^R = [(\alpha+3, -\alpha+4); (2\alpha+3, -2\alpha+5)]$$

$$\tilde{4}^R = [(\alpha+4, -\alpha+5); (2\alpha+4, -2\alpha+6)]$$

Solution 1 *For player* 1 *: the Upper - Upper Game (UUG)*

$$G_{1_\alpha}{}^{UU} = 3_\alpha{}^{UU} u_\alpha{}^{UU} v_\alpha{}^{UU} - 1_\alpha{}^{UU}$$

S.T.

$$1_\alpha{}^{UU} u_\alpha{}^{UU} + 1_\alpha{}^{UU} v_\alpha{}^{UU} - 2_\alpha{}^{UU} \geq 0$$

Then

$$G_{1_\alpha}{}^{UU} = (-2\alpha+5) u_\alpha{}^{UU} v_\alpha{}^{UU} - (-2\alpha+3)$$

S.T.

$$(-2\alpha+3) u_\alpha{}^{UU} + (-2\alpha+3) v_\alpha{}^{UU} - (-2\alpha+4) \geq 0$$

The Lagrange function is given as

$$L_{1_\alpha}{}^{UU}[\hat{x}_\alpha{}^{UU}, \hat{u}_\alpha{}^{UU}, \lambda_{1_\alpha}{}^{UU}, \mu_{1_\alpha}{}^{UU}] = (-2\alpha+5) u_\alpha{}^{UU} v_\alpha{}^{UU} - (-2\alpha+3)$$
$$-\mu_{1_\alpha}{}^{UU}(-2\alpha+3) u_\alpha{}^{UU} + (-2\alpha+3) v_\alpha{}^{UU} - (-2\alpha+4)$$

Applying Theorem 3.1

$$\frac{\partial L_{1_\alpha}{}^{UU}}{\partial u_\alpha{}^{UU}} = (-2\alpha+5) v_\alpha{}^{UU} - \mu_{1_\alpha}{}^{UU}(-2\alpha+3) = 0 \qquad (57)$$

$$\frac{\partial L_{1_\alpha}{}^{UU}}{\partial v_\alpha{}^{UU}} = (-2\alpha+5)u_\alpha{}^{UU} - \mu_{1_\alpha}{}^{UU}(-2\alpha+3) = 0 \quad (58)$$

$$\mu_{1_\alpha}{}^{UU}(-2\alpha+3)u_\alpha{}^{UU} + (-2\alpha+3)v_\alpha{}^{UU} - (-2\alpha+4) = 0 \quad (59)$$

From 57 and 58

$$u_\alpha{}^{UU} = v_\alpha{}^{UU} \quad (60)$$

Since $\mu_{1_\alpha}{}^{UU} > 0$ then from 59 $\Rightarrow$

$$(-2\alpha+3)u_\alpha{}^{UU} + (-2\alpha+3)v_\alpha{}^{UU} - (-2\alpha+4) = 0 \quad (61)$$

From 60 in 61

$$u_\alpha{}^{UU} = v_\alpha{}^{UU} = \frac{-\alpha+2}{-2\alpha+3}$$

By testing this point, the LUG,ULG and LLG are not achieved.So the game for player 1 does not have a surely optimal solution and it will be resolved at the other games.

The Lower - Upper Game (LUG)

$$G_{1_\alpha}{}^{LU} = 3_\alpha{}^{LU}u_\alpha{}^{LU}v_\alpha{}^{LU} - 1_\alpha{}^{LU}$$

S.T.

$$1_\alpha{}^{LU}u_\alpha{}^{LU} + 1_\alpha{}^{LU}v_\alpha{}^{LU} - 2_\alpha{}^{LU} \geq 0$$

Then

$$G_{1_\alpha}{}^{LU} = (2\alpha+3)u_\alpha{}^{LU}v_\alpha{}^{LU} - (2\alpha+1)$$

S.T.

$$(2\alpha+1)u_\alpha{}^{LU} + (2\alpha+1)v_\alpha{}^{LU} - (2\alpha+2) \geq 0$$

The Lagrange function is given as

$$L_{1_\alpha}{}^{LU}[\hat{x}_\alpha{}^{LU}, \hat{u}_\alpha{}^{LU}, \lambda_{1_\alpha}{}^{LU}, \mu_{1_\alpha}{}^{LU}] = (2\alpha+3)u_\alpha{}^{LU}v_\alpha{}^{LU} - (2\alpha+1)$$
$$-\mu_{1_\alpha}{}^{LU}(2\alpha+1)u_\alpha{}^{LU} + (2\alpha+1)v_\alpha{}^{LU} - (2\alpha+2)$$

Applying necessary conditions

$$\frac{\partial L_{1_\alpha}{}^{LU}}{\partial u_\alpha{}^{LU}} = (2\alpha+3)v_\alpha{}^{LU} - \mu_{1_\alpha}{}^{LU}(2\alpha+1) = 0 \quad (62)$$

$$\frac{\partial L_{1_\alpha}{}^{LU}}{\partial v_\alpha{}^{LU}} = (2\alpha+3)u_\alpha{}^{LU} - \mu_{1_\alpha}{}^{LU}(2\alpha+1) = 0 \quad (63)$$

$$\mu_{1_\alpha}{}^{LU}(2\alpha+1)u_\alpha{}^{LU}+(2\alpha+1)v_\alpha{}^{LU}-(2\alpha+2)=0 \tag{64}$$

From 62 and 63

$$u_\alpha{}^{LU}=v_\alpha{}^{LU} \tag{65}$$

Since $\mu_{1_\alpha}{}^{LU}>0$ then from 64 $\Rightarrow$

$$(2\alpha+1)u_\alpha{}^{LU}+(2\alpha+1)v_\alpha{}^{LU}-(2\alpha+2)=0 \tag{66}$$

From 65 in 66

$$u_\alpha{}^{LU}=v_\alpha{}^{LU}=\frac{\alpha+1}{2\alpha+1}$$

The Upper - Lower Game (ULG)

$$G_{1_\alpha}{}^{UL}=3_\alpha{}^{UL}u_\alpha{}^{UL}v_\alpha{}^{UL}-1_\alpha{}^{UL}$$

S.T.

$$1_\alpha{}^{UL}u_\alpha{}^{UL}+1_\alpha{}^{UL}v_\alpha{}^{UL}-2_\alpha{}^{UL}\geq 0$$

Then

$$G_{1_\alpha}{}^{UL}=(-\alpha+4)u_\alpha{}^{UL}v_\alpha{}^{UL}-(-\alpha+2)$$

S.T.

$$(-\alpha+2)u_\alpha{}^{LU}+(-\alpha+2)v_\alpha{}^{UL}-(-\alpha+3)\geq 0$$

The Lagrange function is given as

$$L_{1_\alpha}{}^{UL}[\hat{x}_\alpha{}^{UL},\hat{u}_\alpha{}^{UL},\lambda_{1_\alpha}{}^{UL},\mu_{1_\alpha}{}^{UL}]=(-\alpha+4)u_\alpha{}^{UL}v_\alpha{}^{UL}-(-\alpha+2)$$
$$-\mu_{1_\alpha}{}^{UL}(-\alpha+2)u_\alpha{}^{LU}+(-\alpha+2)v_\alpha{}^{LU}-(-\alpha+3)$$

Applying necessary conditions

$$\frac{\partial L_{1_\alpha}{}^{UL}}{\partial u_\alpha{}^{UL}}=(-\alpha+4)v_\alpha{}^{UL}-\mu_{1_\alpha}{}^{UL}(-\alpha+2)=0 \tag{67}$$

$$\frac{\partial L_{1_\alpha}{}^{UL}}{\partial v_\alpha{}^{UL}}=(-\alpha+4)u_\alpha{}^{UL}-\mu_{1_\alpha}{}^{UL}(-\alpha+2)=0 \tag{68}$$

$$\mu_{1_\alpha}{}^{UL}(-\alpha+2)u_\alpha{}^{LU}+(-\alpha+2)v_\alpha{}^{LU}-(-\alpha+3)=0 \tag{69}$$

From 67 and 68

$$u_\alpha{}^{UL}=v_\alpha{}^{UL} \tag{70}$$

Since ${\mu_1}_\alpha^{UL} > 0$ then from 69 ⇒

$$(-\alpha+2)u_\alpha^{LU} + (-\alpha+2)v_\alpha^{LU} - (-\alpha+3) = 0 \tag{71}$$

From 70 in 71

$$u_\alpha^{UL} = v_\alpha^{UL} = \frac{-\alpha+3}{2(-\alpha+2)}$$

The Lower - Lower Game (LLG)

$${G_1}_\alpha^{LL} = 3_\alpha^{LL} u_\alpha^{LL} v_\alpha^{LL} - 1_\alpha^{LL}$$

S.T.

$$1_\alpha^{LL} u_\alpha^{LL} + 1_\alpha^{LL} v_\alpha^{LL} - 2_\alpha^{LL} \geq 0$$

Then

$${G_1}_\alpha^{LL} = (\alpha+3)u_\alpha^{LL} v_\alpha^{LL} - (\alpha+1)$$

S.T.

$$(\alpha+1)u_\alpha^{LL} + (\alpha+1)v_\alpha^{LL} - (-\alpha+3) \geq 0$$

The Lagrange function is given as

$${L_1}_\alpha^{LL}[\hat{x}_\alpha^{LL}, \hat{u}_\alpha^{LL}, {\lambda_1}_\alpha^{LL}, {\mu_1}_\alpha^{LL}] = (\alpha+3)u_\alpha^{LL} v_\alpha^{LL} - (\alpha+1)$$

$$-{\mu_1}_\alpha^{LL}(\alpha+1)u_\alpha^{LL} + (\alpha+1)v_\alpha^{LL} - (-\alpha+3)$$

Applying necessary conditions

$$\frac{\partial {L_1}_\alpha^{LL}}{\partial u_\alpha^{LL}} = (\alpha+3)v_\alpha^{LL} - {\mu_1}_\alpha^{LL}(\alpha+1) = 0 \tag{72}$$

$$\frac{\partial {L_1}_\alpha^{LL}}{\partial v_\alpha^{LL}} = (\alpha+3)u_\alpha^{LL} - {\mu_1}_\alpha^{LL}(\alpha+1) = 0 \tag{73}$$

$${\mu_1}_\alpha^{LL}(\alpha+1)u_\alpha^{LL} + (\alpha+1)v_\alpha^{LL} - (-\alpha+3) = 0 \tag{74}$$

From 72 and 73

$$u_\alpha^{LL} = v_\alpha^{LL} \tag{75}$$

Since ${\mu_1}_\alpha^{LL} > 0$ then from 74 ⇒

$$(\alpha+1)u_\alpha^{LL} + (\alpha+1)v_\alpha^{LL} - (-\alpha+3) = 0 \tag{76}$$

From 75 in 76

$$u_\alpha{}^{LL} = v_\alpha{}^{LL} = \frac{\alpha+2}{2(\alpha+1)}$$

Similarly, for player 2 : the Upper - Upper Game(UUG)

$$u_\alpha{}^{UU} = \frac{-3\alpha+7}{-2\alpha+3}$$

$$v_\alpha{}^{UU} = \frac{-\alpha+3}{-2\alpha+3}$$

The Lower - Upper Game (LUG)

$$u_\alpha{}^{LU} = \frac{3\alpha+4}{2\alpha+1}$$

$$v_\alpha{}^{LU} = \frac{\alpha+2}{2\alpha+1}$$

The Upper - Lower Game (ULG)

$$u_\alpha{}^{UL} = \frac{-3\alpha+11}{2(\alpha+2)}$$

$$v_\alpha{}^{UL} = \frac{-\alpha+5}{2(-\alpha+2)}$$

The Lower - Lower Game (LLG)

$$u_\alpha{}^{LL} = \frac{3\alpha+8}{2(\alpha+1)}$$

$$v_\alpha{}^{LL} = \frac{\alpha+4}{2(\alpha+1)}$$

6 Conclusion

In this study, we architect (FRCSG), moreover a strategy for tackling such problem is suggested.Firstly, we obtain the lower and upper game then we formulate the α -level that result in dividing game in to four games as follow Upper Upper Game (UUG), Lower Upper Game (LUG), Upper Lower Game (ULG) and Lower Lower Game(LLG).Necessary conditions for determining Nash- Equilibrium points are derived for (UUG).We suggest a strategy to solve such problem by firstly solve the (UUG) if its solution belongs to the (LUG) ,(ULG) and (LLG) then the solution is reached otherwise we solve the other games.A numerical illustration was introduced to clarify the proposed method.

7 References

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